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AN EXTENDED KANTOROVICH METHOD FOR THE SOLUTION OF EIGENVALUE PROBLEMS†

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Abstract—The paper presents an extended Kantorovich method for the solution of eigenvalue problems in partial differential equations. The specific examples treated are: the vibrations of a rectangular membrane and the stability of an elastic rectangular plate compressed in its plane. It is shown that for the membrane problems, the generated expressions for the eigenvalues and eigenfunctions are identical with the corresponding exact solution. For the clamped plate compressed uni-axially or bi-axially, problems which are not separable and for which no exact solutions are available, the generated eigenvalues, based on a one term expression for the eigenfunction, are shown to agree very closely with the relevant results found by other investigators. For plate problems which are separable, the method generates the exact eigenvalues and eigenfunctions. It was found that in all treated cases the final results are independent of the initial choice of the functions and that the iterative procedure converges very rapidly.

INTRODUCTION

IN A recent paper A. D. Kerr [1, 2] extended the Kantorovich method by considering it as only the first step of an iterative procedure. The suggested method was demonstrated on the boundary value problem

$$\nabla^2\Phi(x, y) = -2 \qquad \text{in region } R \tag{1}$$

$$\Phi = 0 \qquad \text{on boundary } B \tag{2}$$

associated with the torsion of a rectangular elastic beam. It was found that the generated one term approximation is independent of the initial choice, that the convergence of the iterative procedure is very rapid, and that the first derivatives of Φ, i.e. the shearing stresses, agree closely with the corresponding values obtained from the exact solution.

Subsequently, A. D. Kerr and H. Alexander [3] used this procedure to solve the boundary value problem

$$\nabla^4 w(x, y) = q/D \qquad \text{in region } R \tag{3}$$

$$w = 0; \qquad \frac{\mathrm{d}w}{\mathrm{d}n} = 0 \qquad \text{on boundary } B \tag{4}$$

associated with the clamped rectangular plate subjected to a uniform lateral load q. Also in this case it was found that the generated one term solution is independent of the initial choice and that the convergence of the iterative procedure is very rapid. The numerical evaluation showed that the obtained deflection surface w agrees very closely throughout the domain with relevant results obtained by other methods. It was also found that the

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accuracy of the obtained second derivatives, i.e. the bending moments, is sufficient for most applications in engineering practice.

The purpose of the present paper is to demonstrate the applicability of the extended Kantorovich method to eigenvalue problems, to show the relative simplicity of the resulting analyses, as well as the high accuracy of the determined eigenvalues.

THE VIBRATIONS OF A RECTANGULAR MEMBRANE

Let us consider, as the first example, the transverse vibrations of a uniformly stretched rectangular membrane which is attached along all four sides to a rigid boundary as shown in Fig. 1. The natural frequencies and natural modes are obtained from the equations (see [4] p. 249).

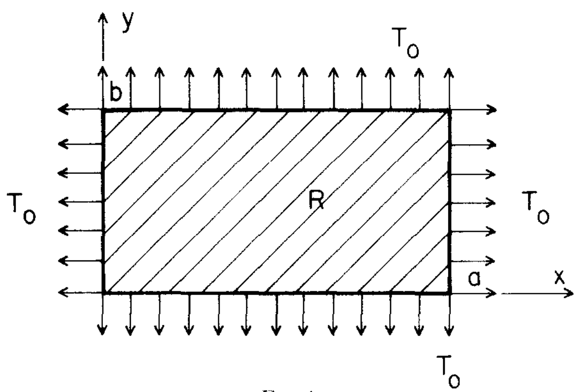

FIG. 1.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{in } R \tag{5}$$

$$u = 0 \qquad \text{on } B \text{ for all } t \tag{6}$$

where $u(x, y, t)$ are the lateral displacements, $a^2 = T_0/\rho$, T_0 is the uniform tension field, and ρ is the mass of the membrane per unit area. Setting $u(x, y, t) = v(x, y)T(t)$ and using the method of separation of variables, we obtain the following eigenvalue problem for $v(x, y)$:

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\lambda v = 0 \qquad \text{in } R \tag{7}$$

$$v = 0 \qquad \text{on } B \tag{8}$$

where $\lambda > 0$.

In the following this eigenvalue problem will be solved by means of the extended Kantorovich method using only a one term approximation.

The equations of the iterative process are derived going out from the Galerkin method. For a one term approximation v, the Galerkin equation may be written as

$$\int_0^a \int_0^b \left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\lambda v\right)\delta v \,\mathrm{d}x\,\mathrm{d}y = 0 \tag{9}$$

assuming that v satisfies the boundary condition (8). According to the extended Kantorovich method the function v is assumed in the form

$$v = v_{ij} = f_i(x)g_j(y) \tag{10}$$

If g_j is given *a priori* then

$$\delta v_{ij} = g_j \delta f_i \tag{11}$$

and equation (9) becomes

$$\int_0^a \left\{\left[\int_0^b g_j^2\,\mathrm{d}y\right]\frac{\mathrm{d}^2 f_i}{\mathrm{d}x^2}+\left[\int_0^b \frac{\mathrm{d}^2 g_i}{\mathrm{d}y^2}g_j\,\mathrm{d}y\right]f_i+\left[\lambda\int_0^b g_j^2\,\mathrm{d}y\right]f_i\right\}\delta f_i\,\mathrm{d}x = 0. \tag{12}$$

The above equation is satisfied when

$$\left[\int_0^b g_j^2\,\mathrm{d}y\right]\frac{\mathrm{d}^2 f_i}{\mathrm{d}x^2}+\left[\int_0^b \frac{\mathrm{d}^2 g_j}{\mathrm{d}y^2}g_j\,\mathrm{d}y\right]f_i+\left[\lambda\int_0^b g_j^2\,\mathrm{d}y\right]f_i = 0. \tag{13}$$

Noting that

$$\int_0^b \frac{\mathrm{d}^2 g_j}{\mathrm{d}y^2}g_j\,\mathrm{d}y = \left(\frac{\mathrm{d}g_j}{\mathrm{d}y}g_j\right)_0^b - \int_0^b \left(\frac{\mathrm{d}g_j}{\mathrm{d}y}\right)^2 \mathrm{d}y \tag{14}$$

and that because of boundary condition (8)

$$g_j(0) = 0; \qquad g_j(b) = 0 \tag{15}$$

and hence the integrated term in (14) vanishes, it follows that equation (13) may be written as

$$\left[\int_0^b g_j^2\,\mathrm{d}y\right]\frac{\mathrm{d}^2 f_i}{\mathrm{d}x^2}+\left[\lambda\int_0^b g_j^2\,\mathrm{d}y-\int_0^b\left(\frac{\mathrm{d}g_j}{\mathrm{d}y}\right)^2\mathrm{d}y\right]f_i = 0. \tag{16}$$

Similarly, if $f_i(x)$ is prescribed then

$$\delta v_{ij} = f_i \delta g_j \tag{17}$$

and equation (9) is satisfied, when

$$\left[\int_0^a f_i^2\,\mathrm{d}x\right]\frac{\mathrm{d}^2 g_j}{\mathrm{d}y^2}+\left[\lambda\int_0^a f_i^2\,\mathrm{d}x-\int_0^a\left(\frac{\mathrm{d}f_i}{\mathrm{d}x}\right)^2\mathrm{d}x\right]g_j = 0. \tag{18}$$

Equations (16) and (18) are the ordinary differential equations of the iterative procedure.

As a first specific example let us extend the problem presented by L. V. Kantorovich and V. I. Krylov [5]. As initial choice we assume

$$v_{10} = f_1(x)g_0(y) = f_1 y(y-b) \tag{19}$$

which satisfies the boundary conditions in (15). With this assumption equation (16) becomes

$$b^2\frac{d^2f_1}{dx^2}+(\lambda b^2-10)f_1 = 0. \tag{20}$$

Assuming that

$$\lambda b^2 > 10 \tag{21}$$

the general solution of equation (20) is

$$f_1(x) = A_{11}\sin\left[\left(\lambda-\frac{10}{b^2}\right)^{\frac{1}{2}}x\right]+A_{21}\cos\left[\left(\lambda-\frac{10}{b^2}\right)^{\frac{1}{2}}x\right]. \tag{22}$$

Boundary condition (8) yields the following conditions on f_1

$$f_1(0) = 0; \qquad f_1(a) = 0. \tag{23}$$

From the first condition it follows that $A_{21} = 0$. The second condition yields

$$A_{11}\sin\left[\left(\lambda-\frac{10}{b^2}\right)^{\frac{1}{2}}a\right] = 0. \tag{24}$$

For a nontrivial solution to exist $A_{11} \neq 0$, and hence

$$\left(\lambda-\frac{10}{b^2}\right)^{\frac{1}{2}}a = n\pi \qquad n = 1, 2, \ldots . \tag{25}$$

Thus the first approximation for the eigenvalue is

$$\lambda_{10} = \left(\frac{n\pi}{a}\right)^2+\frac{10}{b^2} \tag{26}$$

and the corresponding eigenfunction is

$$v_{10} = A_{11}\sin\left(\frac{n\pi x}{a}\right)y(y-b). \tag{27}$$

These are the results of the Kantorovich method.

We proceed now with the iterative procedure setting, in view of equation (27),

$$v_{11} = f_1(x)g_1(y) = \sin\left(\frac{n\pi x}{a}\right)g_1(y). \tag{28}$$

Substituting the above expression in equation (18), it becomes

$$\frac{d^2g_1}{dy^2}+\left(\lambda-\frac{n^2\pi^2}{a^2}\right)g_1 = 0. \tag{29}$$

Assuming that

$$\lambda > \frac{n^2\pi^2}{a^2} \tag{30}$$

the general solution of equation (29) is

$$g_1(y) = A'_{11} \sin\left[\left(\lambda - \frac{n^2\pi^2}{a^2}\right)^{\frac{1}{2}} y\right] + A'_{21} \cos\left[\left(\lambda - \frac{n^2\pi^2}{a^2}\right)^{\frac{1}{2}} y\right]. \tag{31}$$

Substitution of equation (31) into the boundary conditions given in (15) and the subsequent use of the condition for the existence of a nontrivial solution, yields the second approximation for the eigenvalue

$$\lambda_{11} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b^2}\right)^2 \qquad \begin{matrix} n = 1, 2, \ldots \\ m = 1, 2, \ldots \end{matrix} \tag{32}$$

and the corresponding eigenfunction

$$v_{11} = A'_{11} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right). \tag{33}$$

It may be easily shown, by continuing the iteration procedure with

$$v_{21} = f_2(x) g_1(y) = f_2 \sin\left(\frac{m\pi y}{b}\right) \tag{34}$$

that the eigenvalue in (32) and the corresponding eigenfunction in (33) are the final expressions that this procedure generates. A comparison reveals that the obtained results are identical with the exact solution. Thus, for the problem under consideration the method generates the exact solution.

With reference to assumptions (21) and (30) it may be easily shown that when

$$\lambda \leq \frac{10}{b^2} \quad \text{or} \quad \lambda \leq \frac{n^2\pi^2}{a^2} \tag{35}$$

the presented method does not yield any eigenvalues. Thus the obtained eigenvalues are the only ones which this procedure generates.

As a second example let us modify the problem treated above by assuming that the edge along $x = a$ is free. The formulation of this problem is the same as before, except that now the boundary condition along $x = a$ is

$$\left[\frac{\partial u}{\partial x}\right]_{(a,y,t)} = 0. \tag{36}$$

The resulting eigenvalue problem for $v(x, y)$ is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \lambda v = 0 \qquad \text{in } R \tag{37}$$

$$\left.\begin{matrix} v(0, y) = 0; & \left[\dfrac{\partial v}{\partial x}\right]_{(a,y)} = 0 \\ v(x, 0) = 0; & v(x, b) = 0 \end{matrix}\right\}. \tag{38}$$

Proceeding as before it may be shown that the iterative equations (16) and (18) are also valid for the present problem. The first step of the iterative procedure, with v_{10} as given in (19), yields in view of the different boundary conditions on f_1

$$f_1(0) = 0; \qquad \left[\frac{\mathrm{d}f_1}{\mathrm{d}x}\right]_{x=a} = 0 \tag{39}$$

the eigenvalue

$$\lambda_{10} = \left[\frac{(2n-1)\pi}{2a}\right]^2 + \frac{10}{b^2} \qquad n = 1, 2, \ldots \tag{40}$$

and the corresponding eigenfunction

$$v_{10} = A_{11} \sin\left[\frac{(2n-1)\pi x}{2a}\right] y(y-b). \tag{41}$$

The second step with

$$v_{11} = f_1(x) g_1(y) = \sin\left[\frac{(2n-1)\pi x}{2a}\right] g_1(y) \tag{42}$$

yields the eigenvalue

$$\lambda_{11} = \left[\frac{(2n-1)\pi}{2a}\right]^2 + \left(\frac{m\pi}{b}\right)^2 \qquad \begin{array}{l} n = 1, 2, \ldots \\ m = 1, 2, \ldots \end{array} \tag{43}$$

with the corresponding eigenfunction

$$v_{11} = A'_{11} \sin\left[\frac{(2n-1)\pi x}{2a}\right] \sin\left(\frac{m\pi y}{b}\right). \tag{44}$$

It may be shown by continuing the iterations that the eigenvalue in (43) and the eigenfunction in (44) are the final expressions that this procedure generates. A comparison reveals that the generated results are, also for the present problem, identical with the exact solution.

THE STABILITY OF A RECTANGULAR PLATE

In order to study the use of the extended Kantorovich method to a more difficult eigenvalue problem, let us analyze the stability of an elastic clamped rectangular plate subjected in its plane to a constant compression field, N, as shown in Fig. 2.

The pressure force at which the onset of buckling takes place, N_{cr}, is determined (see for example Ref. [6]) from the eigenvalue problem consisting of the differential equation

$$\nabla^4 w + \lambda^2 \nabla^2 w = 0 \qquad \text{in } R \tag{45}$$

and the boundary conditions

$$\left.\begin{array}{r} w = 0 \\ \dfrac{\partial w}{\partial x} = 0 \end{array}\right\} \begin{array}{l} x = \pm a \\ -b \leq y \leq +b \end{array} \tag{46}$$

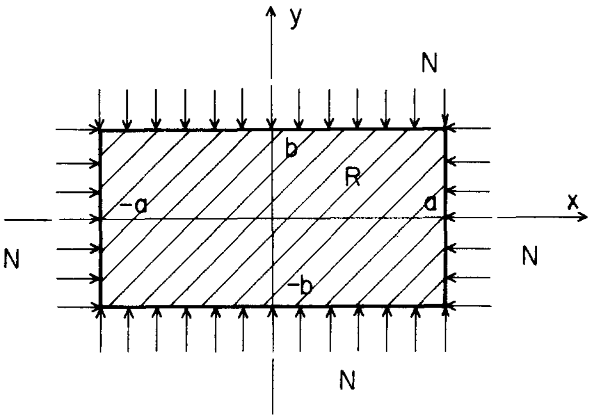

FIG. 2.

and

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial y} &= 0 \end{aligned}\right\} \begin{aligned} &y = \pm b \\ &-a \le x \le +a \end{aligned} \tag{47}$$

where $w(x, y)$ is the lateral deflection,

$$\lambda^2 = \frac{N}{D} \tag{48}$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{49}$$

$$\nabla^4 = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) \tag{50}$$

and D is the flexural rigidity of the plate.

The equations for the iterative procedure are derived going out from the Galerkin method. For a one term approximation w, the Galerkin equation may be written as

$$\int_{-a}^{+a} \int_{-b}^{+b} (\nabla^4 w + \lambda^2 \nabla^2 w) \delta w \, dx \, dy = 0 \tag{51}$$

assuming that w satisfies the boundary conditions given in (46) and (47).

According to the extended Kantorovich method we assume

$$w(x, y) = w_{ij}(x, y) = f_i(x) g_j(y) \tag{52}$$

If $g_j(y)$ is prescribed *a priori* then, proceeding as before, it follows that equation (51) is satisfied when

$$\begin{aligned} &\left[\int_{-b}^{+b} g_j^2 \, dy\right] \frac{d^4 f_i}{dx^4} + \left[\lambda^2 \int_{-b}^{+b} g_j^2 \, dy - 2 \int_{-b}^{+b} \left(\frac{dg_j}{dy}\right)^2 dy\right] \frac{d^2 f_i}{dx^2} \\ &\quad - \left[\lambda^2 \int_{-b}^{+b} \left(\frac{dg_j}{dy}\right)^2 dy - \int_{-b}^{+b} \left(\frac{d^2 g_j}{dy^2}\right)^2 dy\right] f_i = 0. \end{aligned} \tag{53}$$

Similarly, if $f_i(x)$ is prescribed, equation (51) is satisfied when

$$\left[\int_{-a}^{+a} f_i^2\,\mathrm{d}x\right]\frac{\mathrm{d}^4 g_j}{\mathrm{d}y^4}+\left[\lambda^2\int_{-a}^{+a} f_i^2\,\mathrm{d}x-2\int_{-a}^{+a}\left(\frac{\mathrm{d}f_i}{\mathrm{d}x}\right)^2\mathrm{d}x\right]\frac{\mathrm{d}^2 g_j}{\mathrm{d}y^2}$$
$$-\left[\lambda^2\int_{-a}^{+a}\left(\frac{\mathrm{d}f_i}{\mathrm{d}x}\right)^2\mathrm{d}x-\int_{-a}^{+a}\left(\frac{\mathrm{d}^2 f_i}{\mathrm{d}x^2}\right)^2\right]g_j = 0 \tag{54}$$

Equations (53) and (54) are the two ordinary differential equations of the iterative procedure. They will be used in the following to determine the eigenvalues and eigenfunctions of the problem under consideration.

As a first approximation we choose, in accordance with (52),

$$w_{10} = f_1(x)g_0(y) = f_1(x)(y^2-b^2)^2 \tag{55}$$

which satisfies the boundary conditions stated in (47). Substituting equation (55) into equation (53) we obtain, after performing the integrations and dividing the resulting equation by b^5,

$$\left[b^4\frac{256}{315}\right]\frac{\mathrm{d}^4 f_1}{\mathrm{d}x^4}+b^2\left[(\lambda b)^2\frac{256}{315}-2\frac{256}{105}\right]\frac{\mathrm{d}^2 f_1}{\mathrm{d}x^2}-\left[(\lambda b)^2\frac{256}{105}-\frac{128}{5}\right]f_1 = 0 \tag{56}$$

Equation (56) is a fourth order ordinary differential equation with constant coefficients. Its solution consists of four linearly independent functions e^{mx}. The four values of m are obtained from the algebraic equation

$$b^4m^4+b^2\left[(\lambda b)^2-6\right]m^2-\left[3(\lambda b)^2-\frac{63}{2}\right] = 0 \tag{57}$$

Since the magnitude of (λb) is not known, the roots may be real, imaginary, or complex.

Let us assume first, that equation (57) yields two real and two imaginary roots

$$m_{1,2} = \pm\frac{\rho_1}{a};\quad m_{3,4} = \pm i\frac{\varkappa_1}{a} \tag{58}$$

This will be the case when

$$(\lambda b)^2 > \frac{21}{2} \tag{59}$$

The corresponding solution of equation (56) is

$$f_1(x) = A_{11}\sinh\left(\rho_1\frac{x}{a}\right)+A_{12}\cosh\left(\rho_1\frac{x}{a}\right)+A_{13}\sin\left(\varkappa_1\frac{x}{a}\right)+A_{14}\cos\left(\varkappa_1\frac{x}{a}\right) \tag{60}$$

where

$$\left.\begin{matrix}\rho_1\\ \varkappa_1\end{matrix}\right\} = \frac{a}{b}\left\{\left(\left[\frac{(\lambda b)^2}{2}-3\right]^2+\left[3(\lambda b)^2-\frac{63}{2}\right]\right)^{\frac{1}{2}}\mp\left[\frac{(\lambda b)^2}{2}-3\right]\right\}^{\frac{1}{2}} \tag{61}$$

In order to simplify the presentation, the following analysis will be restricted to the determination of eigenvalues whose eigenfunctions are symmetrical with respect to the x and y axes. Assuming the position of the coordinate axes as shown in Fig. 2, the determination of N_{cr} for a square plate becomes a special case of this analysis.

For these cases, we may write

$$f_1(x) = A_{12} \cosh\left(\rho_1 \frac{x}{a}\right) + A_{14} \cos\left(\varkappa_1 \frac{x}{a}\right) \tag{62}$$

Substituting equation (62) into the boundary conditions stated in (46), or into the equivalent ones

$$f_1(\pm a) = 0; \qquad \left(\frac{\mathrm{d}f_1}{\mathrm{d}x}\right)_{\pm a} = 0 \tag{63}$$

we obtain

$$\left.\begin{aligned} A_{12} \cosh \rho_1 + A_{14} \cos \varkappa_1 &= 0 \\ A_{12}\rho_1 \sinh \rho_1 - A_{14}\varkappa_1 \sin \varkappa_1 &= 0 \end{aligned}\right\} \tag{64}$$

The condition for the existence of a non trivial solution is

$$\Delta = 0 \tag{65}$$

where Δ is the determinant of the coefficients. Equation (65) yields

$$\varkappa_1 \operatorname{tg} \varkappa_1 = -\rho_1 \operatorname{tgh} \rho_1 \tag{66}$$

For $a = b$, the roots of equation (66) are shown in Fig. 3. They are

$$(\lambda b)_{10} = 13{\cdot}2932,\ 40{\cdot}0007, \ldots. \tag{67}$$

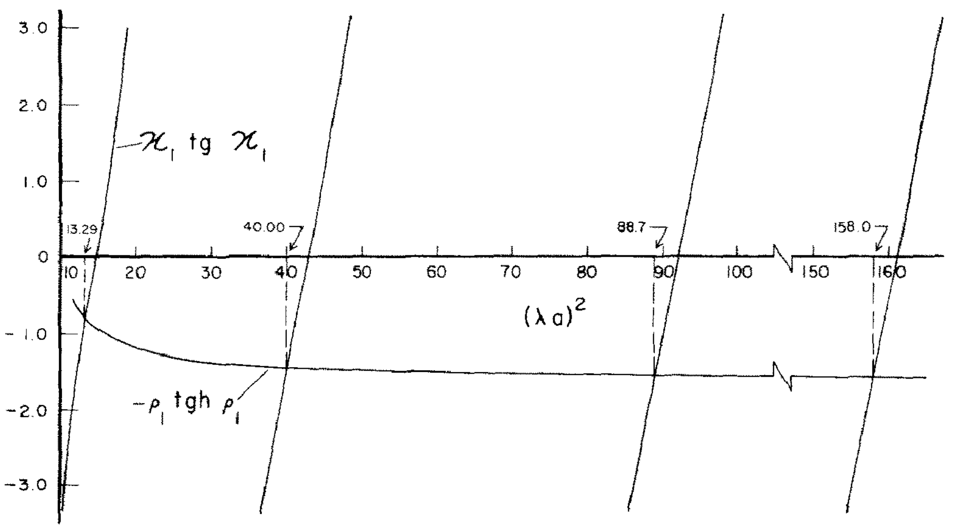

FIG. 3.

For the determination of N_{cr} for the square plate, only the first root is of interest. According to equation (61) the corresponding values ρ_1 and $\varkappa_1$ are

$$\rho_1 = 1{\cdot}0046; \qquad \varkappa_1 = 2{\cdot}8814. \tag{68}$$

Determining A_{14} from the first of the equations in (64), we may write

$$w_{10} = A_{10}\left[\cos\varkappa_1 \cosh\left(\rho_1 \frac{x}{a}\right) - \cosh\rho_1 \cos\left(\varkappa_1 \frac{x}{a}\right)\right](y^2-b^2)^2 \tag{69}$$

where A_{10} is an undetermined constant.

The obtained results are based on the assumption that equation (57) yields two real and two imaginary roots as presented in (58). It should be noted that among the roots which may exist, these are the only ones which yield eigenvalues for the problem under consideration.

The next step in the iterative procedure is to assume

$$w_{11} = f_1(x)g_1(y) = \left[\cos\varkappa_1 \cosh\left(\rho_1 \frac{x}{a}\right) - \cosh\rho_1 \cos\left(\varkappa_1 \frac{x}{a}\right)\right]g_1(y). \tag{70}$$

Equation (54) may be written as

$$\left[a^4 B'_{10}\right]\frac{d^4 g_1}{dy^4} + a^2\left[(\lambda a)^2 B'_{10} - 2B'_{11}\right]\frac{d^2 g_1}{dy^2} - [(\lambda a)^2 B'_{11} - B'_{12}]g_1 = 0 \tag{71}$$

where

$$B'_{10} = \frac{1}{a}\int_{-a}^{+a} f_1^2\,dx; \qquad B'_{11} = a\int_{-a}^{+a}\left(\frac{df_1}{dx}\right)^2 dx; \qquad B'_{12} = a^3\int_{-a}^{+a}\left(\frac{d^2 f_1}{dx^2}\right)^2 dx. \tag{72}$$

Noting that equation (71) is, except for the magnitude of the coefficients, of the same type as equation (56), we assume as before, that the corresponding characteristic equation yields two real and two imaginary roots.

$$\pm\frac{\rho'_1}{b}; \qquad \pm i\frac{\varkappa'_1}{b}. \tag{73}$$

This will be the case when $(\lambda a)^2 > B'_{12}/B'_{11}$.

Proceeding with the iterations as done before it can be shown that, after performing the initial iteration (the Kantorovich Method), the analysis reduces to the following iterative scheme with $n = 1$ as starting index:

For $n = 1$ determine

$$B'_{n0}, \qquad B'_{n1}, \qquad B'_{n2} \tag{74}$$

and then noting that

$$\left.\begin{matrix}\rho'_n \\ \varkappa'_n\end{matrix}\right\} = \frac{b}{a}\left\{\left(\left[\frac{(\lambda a)^2}{2} - \frac{B'_{n1}}{B'_{n0}}\right]^2 + \left[\frac{B'_{n1}}{B'_{n0}}(\lambda a)^2 - \frac{B'_{n2}}{B'_{n0}}\right]\right)^{\frac{1}{2}} \mp \left[\frac{(\lambda a)^2}{2} - \frac{B'_{n1}}{B'_{n0}}\right]\right\}^{\frac{1}{2}} \tag{75}$$

and

$$(\lambda a)^2 > \frac{B'_{n2}}{B'_{n1}} \tag{76}$$

find the roots of

$$\varkappa'_n \operatorname{tg}\varkappa'_n = -\rho'_n \operatorname{tgh}\rho'_n. \tag{77}$$

With the obtained values for ρ_n', $\varkappa_n'$ determine

$$B_{(n+1)0}, \qquad B_{(n+1)1}, \qquad B_{(n+1)2} \tag{78}$$

and then noting that

$$\left.\begin{matrix}\rho_{n+1}\\ \varkappa_{n+1}\end{matrix}\right\} = \frac{a}{b}\left\{\left(\left[\frac{(\lambda b)^2}{2} - \frac{B_{(n+1)1}}{B_{(n+1)0}}\right]^2 + \left[\frac{B_{(n+1)1}}{B_{(n+1)0}}(\lambda b)^2 - \frac{B_{(n+1)2}}{B_{(n+1)0}}\right]\right)^{\frac{1}{2}} \mp \left[\frac{(\lambda b)^2}{2} - \frac{B_{(n+1)1}}{B_{(n+1)0}}\right]\right\}^{\frac{1}{2}} \tag{79}$$

and

$$(\lambda b)^2 > \frac{B_{(n+1)2}}{B_{(n+1)1}} \tag{80}$$

find the roots of

$$\varkappa_{n+1}\,\mathrm{tg}\,\varkappa_{n+1} = -\rho_{n+1}\,\mathrm{tgh}\,\rho_{n+1}. \tag{81}$$

Then, start next cycle by substituting $n = 2$ in (74) and so on. The B' values are

$$\left.\begin{aligned}
B_{n0}' &= \frac{1}{a}\int_{-a}^{+a} f_n^2\,\mathrm{d}x = \left[1 + \frac{(\varkappa_n^2 - 3\rho_n^2)}{2\rho_n(\rho_n^2+\varkappa_n^2)}\sinh 2\rho_n\right]\cos^2\varkappa_n \\
&\quad + \left[1 + \frac{(\rho_n^2 - 3\varkappa_n^2)}{2\varkappa_n(\rho_n^2+\varkappa_n^2)}\sin 2\varkappa_n\right]\cosh^2\rho_n \\
B_{n1}' &= a\int_{-a}^{+a}\left(\frac{\mathrm{d}f_n}{\mathrm{d}x}\right)^2\mathrm{d}x = -\rho_n^2\left[1 - \frac{(\rho_n^2 - 3\varkappa_n^2)}{2\rho_n(\rho_n^2+\varkappa_n^2)}\sinh 2\rho_n\right]\cos^2\varkappa_n \\
&\quad + \varkappa_n^2\left[1 - \frac{(\varkappa_n^2 - 3\rho_n^2)}{2\varkappa_n(\rho_n^2+\varkappa_n^2)}\sin 2\varkappa_n\right]\cosh^2\rho_n \\
B_{n2}' &= a^3\int_{-a}^{+a}\left(\frac{\mathrm{d}^2f_n}{\mathrm{d}x^2}\right)^2\mathrm{d}x = \rho_n^4\left[1 + \frac{(\rho_n^2 + 5\varkappa_n^2)}{2\rho_n(\rho_n^2+\varkappa_n^2)}\sinh 2\rho_n\right]\cos^2\varkappa_n \\
&\quad + \varkappa_n^4\left[1 + \frac{(\varkappa_n^2 + 5\rho_n^2)}{2\varkappa_n(\rho_n^2+\varkappa_n^2)}\sin 2\varkappa_n\right]\cosh^2\rho_n.
\end{aligned}\right\} \tag{82}$$

The coefficients

$$\left.\begin{aligned}
B_{(n+1)0} &= \frac{1}{b}\int_{-b}^{+b} g_n^2\,\mathrm{d}y; \qquad B_{(n+1)1} = b\int_{-b}^{+b}\left(\frac{\mathrm{d}g_n}{\mathrm{d}y}\right)^2\mathrm{d}y \\
B_{(n+1)2} &= b^3\int_{-b}^{+b}\left(\frac{\mathrm{d}^2g_n}{\mathrm{d}y^2}\right)^2\mathrm{d}y
\end{aligned}\right\} \tag{83}$$

may be obtained from (82) by the following substitution

$$\left.\begin{aligned}
B_{n0}', B_{n1}', B_{n2}' &\rightarrow B_{(n+1)0}, B_{(n+1)1}, B_{(n+1)2} \\
\rho_n, \varkappa_n &\rightarrow \rho_n', \varkappa_n'.
\end{aligned}\right\} \tag{84}$$

In order to study the convergence of the iterative procedure, the problem was numerically evaluated when the domain under consideration is a square, $a = b$. For this case $\rho_\infty = \rho'_\infty$ and $\varkappa_\infty = \varkappa'_\infty$. For each step of the procedure, the first eigenvalue was determined and the results are presented in the following table:

n	Kantorovich method 0	1	2	3
$(\lambda a)^2_{nn}$	—	13·1145	13·1138	13·1138
ρ'_n	—	1·13918	1·13319	1·13318
$\varkappa'_n$	—	2·82430	2·82687	2·82687
$(\lambda a)^2_{(n+1)n}$	13·2932	13·1138	13·1138	13·1138
ρ_{n+1}	1·00464	1·13289	1·13318	1·13318
$\varkappa_{n+1}$	2·88141	2·82699	2·82687	2·82687

It can be seen that the convergence is very rapid and that within the accuracy of the presented calculations already the third step yields the final magnitude of the eigenvalue. Noting equation (48), it follows that the critical buckling load for a square plate is

$$N_{cr} = 13{\cdot}1138\frac{D}{a^2} \tag{85}$$

or rewritten

$$N_{cr} = 5{\cdot}3148\frac{\pi^2 D}{(2a)^2}. \tag{86}$$

The corresponding buckling mode is

$$\begin{aligned} w_\infty = A_0\left[\cos\varkappa_\infty\cosh\left(\rho_\infty\frac{x}{a}\right) - \cosh\rho_\infty\cos\left(\varkappa_\infty\frac{x}{a}\right)\right] \\ \times\left[\cos\varkappa_\infty\cosh\left(\rho_\infty\frac{y}{b}\right) - \cosh\rho_\infty\cos\left(\varkappa_\infty\frac{y}{b}\right)\right]. \end{aligned} \tag{87}$$

where

$$\left.\begin{aligned} \rho_\infty &= 1{\cdot}13318 \\ \varkappa_\infty &= 2{\cdot}82687 \end{aligned}\right\} \tag{88}$$

and A_0 is an undetermined constant.

It is of interest to note that the obtained buckling load agrees very closely with the corresponding result of G. I. Taylor [7] and O. H. Faxén [8], $N_{cr} = 5{\cdot}304\,\pi^2 D/(2a)^2$. The difference is about 0·2 per cent.

As another example, which is also not separable and for which no exact solution is available in the literature, let us determine the critical load of a clamped plate which is compressed uniaxially as shown in Fig. 4.

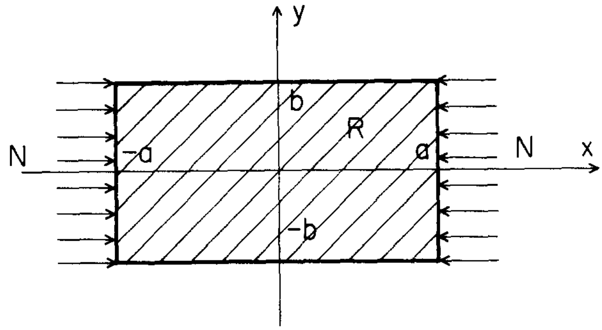

FIG. 4.

The critical load, N_{cr}, is determined from the eigenvalue problem consisting of the differential equation

$$\nabla^4 w + \frac{N}{D}\frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{in } R \tag{89}$$

and the boundary conditions given in (46) and (47). Restricting $w(x, y)$ to a one term expression and then proceeding with the iterations as done previously, we obtain for a square plate the critical load

$$N_{cr} = 24{\cdot}9129\frac{D}{a^2}. \tag{90}$$

The corresponding value obtained by Faxén [8] and by S. Levy [9] is $N_{cr} = 24{\cdot}86\ D/a^2$. Thus, also for this example, the agreement is very close. The rather small difference of about 0·2 per cent appears indeed negligible in view of Levy's estimate that the possible error in his numerical results is of the order of 0·1 per cent.

When two parallel edges of the plate are simply supported the problem is separable. For these cases the extended Kantorovich method generates, using a one term expression for the function, the exact results obtained by means of the M. Lévy approach.

CONCLUSION

For the treated problems, it was found that the extended Kantorovich method generates, using a one term expression for the eigenfunction, the exact eigenvalues and eigenfunctions where the conventional method of separation of variables is applicable. For the clamped rectangular plate, which is not separable, this method generates a highly accurate first eigenvalue, even when the eigenfunction is restricted to only a one term expression. It was found that in each case the final results are independent of the initial choice of the function and that in all treated cases the iterative procedure converges very rapidly.

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Абстракт—В настоящей работе деатся расширенный метод Канторовича решения задач для собственных значений в дифференциальных уравнениях с частными производными. Рассматриваются особые примеры: вибрация прямоугольной мембраны и устойчивость упругой, прямоугольной пластинки, сжатой в ее плоскости. Оказывается, что в задаче мембраны полученные выражения для собственных значений и собственных функций являются тождественными с соответствующим точным решением. Для случая зашемленной пластинки, сжатой в одном и в двух направлениях, оказывается, что в неразделимых задачах и для которых не получаются точные решения, выведенные собственные значения, основанные на одном члене выражения для собственной функции, сходятся очень близко с соответствующими результатами, полученными другими исследователями. Для отдельных задач пластинки, благодаря представленному методу, получаются точные собственные значения и собственные функции. Констатируется, что во всех рассматриваемых случаях, остаточные результаты не зависят от начального выбора функций. Процесс итерации происходит очень быстро.